Revision Summary of DSE Mathematics (Compulsory Part)
Polynomials

**Like terms and unlike terms**

e.g. \(2x \) and \(6x\) are like terms and \(2x + 6x\) can be simplified as \((2 + 6)x = 8x\)

\(4pq^2\) and \(-pq^2\) are like terms and \(4pq^2 - pq^2\) can be simplified as \((4 - 1)pq^2 = 3pq^2\)

\(6ab\) and \(5ac\) are unlike terms and they cannot be simplified

**Factorization**

1. \(a^2 - b^2 \equiv (a+b)(a-b)\)
2. \(a^2 + 2ab + b^2 \equiv (a+b)^2\)
3. \(a^2 - 2ab + b^2 \equiv (a-b)^2\)
4. \(a^3 + b^3 \equiv (a+b)(a^2 - ab + b^2)\)
5. \(a^3 - b^3 \equiv (a-b)(a^2 + ab + b^2)\)

**Remainder and Factor Theorems**

**Remainder theorem**

When a polynomial \(f(x)\) is divided by a linear polynomial \(ax + b\) where \(a \neq 0\), the remainder is \(f\left(-\frac{b}{a}\right)\).

**Factor theorem**

If \(f(x)\) is a polynomial and \(f\left(-\frac{b}{a}\right) = 0\), then the polynomial \(ax + b\) is a factor of the polynomial \(f(x)\).

Conversely, if a polynomial \(ax + b\) is a factor of polynomial \(f(x)\), then \(f\left(-\frac{b}{a}\right) = 0\).

Let \(f(x) = x^3 + 4x^2 - 3x - 18\).

(a) Find \(f(2)\).

\[f(2) = (2)^3 + 4(2)^2 - 3(2) - 18 = 0\]

(b) Factorize \(f(x)\).

Since \(f(2) = 0\), \(x - 2\) is a factor of \(f(x)\).

\[x^3 + 4x^2 - 3x - 18 = (x - 2)(x^2 + 6x + 9) = (x - 2)(x + 3)^2\]

Let \(f(x) = (x - 3)(2x + 1) + x + 3\). Find the remainder when \(f(x)\) is divided by \(x + 3\).

The required remainder

\[= f(-3) = [(-3) - 3][2(-3) + 1] + (-3) + 3 = -6(-5) = 30\]
Solving Quadratic Equations in One Unknown

**Factor method**
e.g. Solve \( x^2 + 2x - 3 = 0 \).

\[
(x + 3)(x - 1) = 0
\]

\[
x + 3 = 0 \quad \text{or} \quad x - 1 = 0
\]

\[
x = -3 \quad \text{or} \quad x = 1
\]

**Completing the square**

Convert it to the form of \( a(x^2 + 2px + p^2) = q \), \( a(x + p)^2 = q \), \( x = -p \pm \sqrt{\frac{q}{a}} \)

**Quadratic formula**
e.g. Solve \( x^2 - 7x + 9 = 0 \).

Using the quadratic formula,

\[
x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(9)}}{2(1)}
\]

\[
= \frac{7 \pm \sqrt{49 - 36}}{2} = \frac{7 \pm \sqrt{13}}{2}
\]

**Nature of Roots of a Quadratic Equation**
Consider the quadratic equation \( ax^2 + bx + c = 0 \), where \( a \neq 0 \).

<table>
<thead>
<tr>
<th>Discriminant ((\Delta = b^2 - 4ac))</th>
<th>( \Delta &gt; 0 )</th>
<th>( \Delta = 0 )</th>
<th>( \Delta &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nature of roots</td>
<td>2 distinct real roots</td>
<td>1 double real root</td>
<td>no real roots</td>
</tr>
</tbody>
</table>

**Sum and Product of Roots of a Quadratic Equation**
If \( \alpha \) and \( \beta \) are the roots of the quadratic equation \( ax^2 + bx + c = 0 \), where \( a \neq 0 \), then

\[
\text{sum of roots} = \alpha + \beta = -\frac{b}{a} \quad , \quad \text{product of roots} = \alpha \beta = \frac{c}{a}
\]

**Solving Equations Reducible to Quadratic Equations**

**Equations of Higher Degree**
e.g. Solve \( x^4 - 6x^2 - 7 = 0 \).

By substituting \( x^2 = u \) into the equation \( x^4 - 6x^2 - 7 = 0 \), we have

\[
u^2 - 6u - 7 = 0
\]

\[
(u - 7)(u + 1) = 0
\]

\[
u = 7 \quad \text{or} \quad u = -1
\]

Since \( u = x^2 \), we have

\[
\begin{align*}
x^2 &= 7 \quad \text{or} \quad x^2 = -1 \quad \text{(rejected)}
\end{align*}
\]

\[
\text{For any real number} \ x, \ x^2 \geq 0.
\]

\[
x = \pm \sqrt{7}
\]
Number and Algebra

**Fractional Equations**
e.g. Solve \( \frac{2}{x-1} - x = 0 \).

\[
\frac{2}{x-1} - x = 0 \\
2 - x(x-1) = 0 \quad \blacktriangledown \text{Multiply both sides by } x - 1.
\]

\[
2 - x^2 + x = 0 \\
x^2 - x - 2 = 0 \\
(x+1)(x-2) = 0 \\
x = -1 \text{ or } x = 2
\]

**Equations with Square Root Signs**
e.g. Solve \( \sqrt{x} + 2x = 0 \).

\[
\sqrt{x} + 2x = 0 \\
\sqrt{x} = -2x \\
(\sqrt{x})^2 = (-2x)^2 \quad \blacktriangledown \text{Square both sides.}
\]

\[
x = 4x^2 \\
4x^2 - x = 0 \\
x(4x-1) = 0 \\
\]

\[
x = 0 \text{ or } x = \frac{1}{4} \text{ (rejected) } \blacktriangledown
\]

**Exponential Equations**
e.g. Solve \( 2^{2x} - 2^x = 0 \). …..(1)

Substitute \( 2^x = u \) into (1).

\[
u^2 - u = 0 \quad \blacktriangledown 2^{2x} = (2^x)^2 = u^2 \\
u(u-1) = 0 \\
u = 0 \text{ or } u = 1 \\
\]

Since \( 2^x = u \), we have

\[
2^x = 0 \text{ (rejected) or } 2^x = 1 \\
x = 0
\]

**Logarithmic Equations**
e.g. Solve \( \log_3 x^2 - \log_3 x = 0 \). …..(2)

Substitute \( \log_3 x = u \) into (2).

\[
u^2 - u = 0 \quad \blacktriangledown u^2 - u = 0 \\
u(u-1) = 0 \\
u = 0 \text{ or } u = 1 \\
\]

Since \( \log_3 x = u \), we have

\[
\log_3 x = 0 \text{ or } \log_3 x = 1 \\
x = \frac{1}{3} \text{ or } x = 3
\]

**Complex Numbers**
Complex number \( a + bi \), where \( a \) and \( b \) are real numbers and \( i^2 = -1 \), i.e. \( i = \sqrt{-1} \).

\( a + bi \) is purely real if and only if \( b = 0 \).
\( a + bi \) is purely imaginary if and only if \( a = 0 \) and \( b \neq 0 \).

**Addition**
\( (a + bi) + (c + di) = (a + c) + (b + d)i \)

**Subtraction**
\( (a + bi) - (c + di) = (a - c) + (b - d)i \)

**Multiplication**
\( (a + bi)(c + di) = (a + bi)(c + (a + bi)(di)) \)
\[
= (ac - bd) + (ad + bc)i
\]

**Division**
\( \frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i \)

**Equality**
\( a + bi = c + di \) if and only if \( a = c \) and \( b = d \).
Graphs of Quadratic Functions

Features of the graphs of quadratic functions

Consider a quadratic function \( y = ax^2 + bx + c \).

<table>
<thead>
<tr>
<th>( a &gt; 0 )</th>
<th>( a &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Graph of a quadratic function with ( a &gt; 0 )." /></td>
<td><img src="image2" alt="Graph of a quadratic function with ( a &lt; 0 )." /></td>
</tr>
<tr>
<td>The graph opens upwards.</td>
<td>The graph opens downwards.</td>
</tr>
</tbody>
</table>

Finding the Optimum Values of Quadratic Functions

(a) To complete the square for \( x^2 \pm kx \), add \( \left( \frac{k}{2} \right)^2 \). Then

\[
x^2 \pm kx + \left( \frac{k}{2} \right)^2 = x^2 \pm 2 \left( \frac{k}{2} \right) k = \left( x \pm \frac{k}{2} \right)^2
\]

(b) For a quadratic function \( y = a(x - h)^2 + k \),
   (i) if \( a > 0 \), then the minimum value of \( y \) is \( k \) when \( x = h \),
   (ii) if \( a < 0 \), then the maximum value of \( y \) is \( k \) when \( x = h \).

Laws of Indices

For integral indices \( p \) and \( q \), let \( a, b \) be real numbers.
For rational indices \( p \) and \( q \), let \( a, b \) be non–negative numbers.

1. \( a^p \times a^q = a^{p+q} \)
2. \( \frac{a^p}{a^q} = a^{p-q} \), where \( a \neq 0 \)
3. \( (a^p)^q = a^{pq} \)
4. \( (ab)^p = a^p b^p \)
5. \( \left( \frac{a}{b} \right)^p = \frac{a^p}{b^p} \), where \( b \neq 0 \)
6. \( \frac{1}{a^p} = a^{-p} \), where \( a \neq 0 \)
7. \( a^0 = 1 \), where \( a \neq 0 \)
8. \( a^{\frac{p}{q}} = (\sqrt[q]{a})^p = \sqrt[q]{a^p} \) for integers \( p \) and \( q \) where \( a, q > 0 \).
Laws of logarithms

If \( x = a^y, a > 0 \) and \( a \neq 1 \), then \( y = \log_a x \).

For \( M, N, a, b > 0 \) and \( a, b \neq 1 \),

1. \( \log_a 1 = 0 \)
2. \( \log_a a = 1 \)
3. \( \log_a (MN) = \log_a M + \log_a N \)
4. \( \log_a \frac{M}{N} = \log_a M - \log_a N \)
5. \( \log_a (M^k) = k \log_a M \) (\( k \) is any real number)
6. \( a^{\log_a M} = M \)
7. \( \log_a N = \frac{\log_b N}{\log_b a} \), where \( b > 0 \) and \( b \neq 1 \)

Note: \( \log_a x \) is undefined for \( x \leq 0 \).

Graphs of exponential functions and logarithmic functions

Percentages

- new value = initial value \( \times \) (1 + percentage change)
- percentage change = \( \frac{\text{new value} - \text{initial value}}{\text{initial value}} \times 100\% \)

Selling problems

A book is marked at $78 for sale. A customer is offered a 10% discount. Find the selling price.

solving price = marked price \( \times \) (1 – discount percent)

\[ = 78 \times (1 - 10\%) = 70.2 \]

Simple and compound interest

\( A = P + I \)

simple interest \( I = P \times r\% \times n \)

compound interest \( I = P(1 + r\%)^n - P \)

where \( A \): amount, \( P \): principal, \( I \): interest, \( r \): interest rate per period, \( n \): number of periods.

Ratio

Given \( a:b = 3:4 \) and \( b:c = 6:7 \). Find \( a:b:c \).

Since \( L.C.M. \) of 4 and 6 is 12
\( a:b:c = 9:12:14 \)
Variation

Direct variation
\[ y \propto x, \quad y = kx \quad \text{where} \quad k \neq 0. \]

Inverse Variation
\[ y \propto \frac{1}{x}, \quad y = \frac{k}{x} \quad \text{or} \quad xy = k \quad \text{where} \quad k \neq 0 \]

Joint variation
1. \( z \) varies jointly as \( x \) and \( y \)
   \[ z \propto xy, \quad z = kxy \quad \text{where} \quad k \neq 0. \]
2. \( z \) varies directly as \( x \) and inversely as \( y \)
   \[ z \propto \frac{x}{y}, \quad z = \frac{kx}{y} \quad \text{where} \quad k \neq 0. \]

Partial variation
1. \( z \) partly varies as \( x \) and partly varies as \( y \)
   \[ z = k_1 x + k_2 y \quad \text{where} \quad k_1, k_2 \neq 0. \]
2. \( z \) partly constant and partly varies as \( y \)
   \[ z = k_1 + k_2 y \quad \text{where} \quad k_2 \neq 0. \]
3. \( z \) partly varies as \( x \) and partly varies inversely as \( y \)
   \[ z = k_1 x + \frac{k_2}{y} \quad \text{where} \quad k_1, k_2 \neq 0. \]

Errors

absolute error = \( |\text{measured value} - \text{true value}| \)

Note: maximum absolute error = largest possible error, in this case the true value is not known.

relative error = \( \frac{\text{absolute error}}{\text{true value}} \) or \( \frac{\text{maximum absolute error}}{\text{measured value}} \)

percentage error = relative error \( \times 100\% \)

Arithmetic sequence
\[ a, \quad a + d, \quad a + 2d, \quad a + 3d, \ldots \]
Common difference = \( d \). First term \( T(1) = a \)

\( n \)th term \( T(n) = a + (n - 1)d \)

Let the sum of the first \( n \) terms be \( S(n) = T(1) + T(2) + T(3) + \cdots + T(n) \), then
\[ S(n) = n \left( \frac{T(1) + T(n)}{2} \right) \quad \text{or} \quad n \left[ \frac{2a + (n - 1)d}{2} \right]. \]

If \( x, y, z \) form an arithmetic sequence, then \( y = \frac{x + z}{2} \).

Geometric sequence
\[ a, \quad ar, \quad ar^2, \quad ar^3, \ldots \]
Common ratio = \( r \). First term \( T(1) = a \)

\( n \)th term \( T(n) = ar^{n-1} \)

Let the sum of the first \( n \) terms be \( S(n) = T(1) + T(2) + T(3) + \cdots + T(n) \), then
\[ S(n) = \frac{a(r^n - 1)}{r - 1}, \quad r \neq 1 \]

Sum to infinity \( = \frac{a}{1 - r}, \quad -1 < r < 1 \)

If \( x, y, z \) form a geometric sequence, then \( y^2 = xz, \quad i.e. \quad y = \pm \sqrt{xz} \).
The \( n \)th term of an arithmetic sequence is \( \frac{-n}{2} \). Find the first term and the common ratio.

\[
T(1) = \frac{-1}{2}, \quad T(2) = \frac{-2}{2} = -1, \quad d = -1 - \frac{-1}{2} = \frac{-1}{2}.
\]

Express 0.16 as a fraction.

\[
0.16 = 0.1 + 0.06 + 0.006 + 0.0006 + \cdots
\]

\[
= 0.1 + \frac{0.06}{1 - 0.1} = \frac{1}{10} + \frac{6}{90} = \frac{1}{6}
\]

Simultaneous equations in two unknowns

\[
\begin{align*}
Ax + By + C &= 0 \\
y &= ax^2 + bx + c
\end{align*}
\]

By substitution

\[
Ax + B(ax^2 + bx + c) + C = 0
\]

\[
Bax^2 + (A + Bb)x + (Bc + C) = 0
\]

Since \( A, B, C, a, b, c \) are known, \( x \) can be find by quadratic formula.

Having found \( x \), \( y = \frac{-(Ax + C)}{B} \).

By graphical method

The coordinates of intersecting points \((x_1, y_1)\) and \((x_2, y_2)\) represents the solutions.

There may be 0, 1, or 2 intersecting points, depending on the number of solutions.

Inequalities

Rule 1 If \( a < b \) and \( b < c \), then \( a < c \).

Rule 2 Addition to both sides does not change the inequalities sign.

\[
a < b \quad \text{then} \quad a + c < b + c
\]

Rule 3 For \( c > 0, \ a < b \) then \( ac < bc \).

For \( c < 0, \ a < b \) then \( ac > bc \).

Graphical representation
Number and Algebra

Solving \( f(x) > k, f(x) < k, f(x) \geq k \) and \( f(x) \leq k \) graphically:

Step 1 Draw the graph \( y = f(x) \) and \( y = k \).

Intersecting points should be labeled.

Solve \( x^2 + 3x + 2 > 6 \)

Step 2 Identify of \( x \) which \( f(x) > k \).

\( x < -4 \) or \( x > 1 \).

Note: ‘>’ is different from ‘≥’. If the equation is changed to ‘\( f(x) \geq k \)’, then the solution is \( x \leq -4 \) or \( x \geq 1 \).

Solving a Linear Inequality in Two Unknowns Graphically

Linear programming

(a) Solve graphically the system of inequalities:

\[
\begin{align*}
   x + 2y & \geq 2 \\
   x + y & \leq 3 \\
   x & \geq 0
\end{align*}
\]

(b) Hence find the maximum and minimum value of \( f(x, y) = x - 2y \) subject to the above constraints. Consider the intersecting points

<table>
<thead>
<tr>
<th>(x,y)</th>
<th>( f(x, y) = x - 2y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,1)</td>
<td>-2</td>
</tr>
<tr>
<td>(0,3)</td>
<td>-6</td>
</tr>
<tr>
<td>(4,-1)</td>
<td>6</td>
</tr>
</tbody>
</table>

Maximum value = 6 and minimum value = -6

Alternative method:

Draw a dotted line of \( x - 2y = 0 \).

To determine the extremes of \( x - 2y \), move the dotted line vertically and find the two critical positions such that the line is about to leave the shaded region. The value of \( x - 2y \) at the two extreme positions give the maximum and minimum.
Different Numeral Systems

<table>
<thead>
<tr>
<th>Numeral system</th>
<th>Numerals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Denary system</td>
<td>0, 1, 2, 3, 4, 5, 6, 7, 8 and 9</td>
</tr>
<tr>
<td>Binary system</td>
<td>0 and 1</td>
</tr>
<tr>
<td>Hexadecimal system</td>
<td>0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E and F</td>
</tr>
</tbody>
</table>

In a number, the position of each digit has fixed place value.

For example, in 4305, the place value of 3 is $10^3$. In C8A₁₆, the place value of 8 is 16.

The value of a number can be expressed in the expanded form.

For example, $4305_{10} = 4 \times 10^3 + 3 \times 10^2 + 0 \times 10^1 + 5 \times 1$

$= 1 \times 2^4 + 1 \times 2^3 + 2^2 + 0 \times 2^1 + 1 \times 1$

$C8A_{16} = 12 \times 16^2 + 8 \times 16^1 + 10 \times 1$

Converting a binary number or a hexadecimal number into a denary number:

$11101_2 = 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 1 = 29$

$C8A_{16} = 12 \times 16^2 + 8 \times 16^1 + 10 \times 1 = 3210$

Converting a denary number into a binary number or a hexadecimal number:

$29_{10} = 11101_2$$3210_{10} = C8A_{16}$

<table>
<thead>
<tr>
<th>Transformation of the graph</th>
<th>Transformation of the function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Translate upwards by k units</td>
<td>$y = f(x) + k$</td>
</tr>
<tr>
<td>Translate downwards by k units</td>
<td>$y = f(x) = k$</td>
</tr>
<tr>
<td>Translate leftwards by k units</td>
<td>$y = f(x + k)$</td>
</tr>
<tr>
<td>Translate rightwards by k units</td>
<td>$y = f(x - k)$</td>
</tr>
<tr>
<td>Reflect about the x-axis</td>
<td>$y = -f(x)$</td>
</tr>
<tr>
<td>Reflect about the y-axis</td>
<td>$y = f(-x)$</td>
</tr>
<tr>
<td>Enlarge along the y-axis k times the original, where $k &gt; 1$</td>
<td>$y = kf(x)$</td>
</tr>
<tr>
<td>Reduce along the y-axis k times the original, where $0 &lt; k &lt; 1$</td>
<td></td>
</tr>
<tr>
<td>Enlarge along the x-axis $\frac{1}{k}$ times the original, where $k &gt; 1$</td>
<td>$y = f(kx)$</td>
</tr>
<tr>
<td>Reduce along the x-axis $\frac{1}{k}$ times the original, where $0 &lt; k &lt; 1$</td>
<td></td>
</tr>
</tbody>
</table>
### Mensuration of common plane figures and solids

#### Plane figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Perimeter</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trapezium</td>
<td>$a + b + c + d$</td>
<td>$(a+b)h/2$</td>
</tr>
<tr>
<td>Rhombus</td>
<td>$4a$</td>
<td>$xy/2$</td>
</tr>
<tr>
<td>Circle</td>
<td>$2\pi r$</td>
<td>$\pi r^2$</td>
</tr>
<tr>
<td>Sector</td>
<td>$\frac{2\pi r\theta}{360^\circ} + 2r$</td>
<td>$\pi r^2 \times \frac{\theta}{360^\circ}$</td>
</tr>
</tbody>
</table>

#### Solids

<table>
<thead>
<tr>
<th>Figure</th>
<th>Volume</th>
<th>Surface Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube</td>
<td>$a^3$</td>
<td>$6a^2$</td>
</tr>
<tr>
<td>Cuboid</td>
<td>$bhl$</td>
<td>$2(bh + bl + hl)$</td>
</tr>
</tbody>
</table>
### Measures, Shape and Space

<table>
<thead>
<tr>
<th>Geometry</th>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Prism</strong></td>
<td>$Ah$</td>
<td>Base perimeter $\times h + 2A$</td>
</tr>
<tr>
<td><strong>Cylinder</strong></td>
<td>$\pi r^2 h$</td>
<td>$2\pi rh + 2\pi r^2$</td>
</tr>
<tr>
<td><strong>Pyramid</strong></td>
<td>$\frac{1}{3}Ah$</td>
<td>$A + \text{total area of lateral faces}$</td>
</tr>
<tr>
<td><strong>Cone</strong></td>
<td>$\frac{1}{3}\pi r^2 h$</td>
<td>$\pi rl + \pi r^2$</td>
</tr>
<tr>
<td><strong>Sphere</strong></td>
<td>$\frac{4}{3}\pi r^3$</td>
<td>$4\pi r^2$</td>
</tr>
</tbody>
</table>

*Euler’s formula* (for a convex polyhedron only)

$$V - E + F = 2$$

where $V$: number of vertices, $E$: number of edges, $F$: number of faces.
Measures, Shape and Space

Similar plane figures and solids

\[
\left( \frac{l_1}{l_2} \right)^2 = \frac{A_1}{A_2}, \quad \left( \frac{l_1}{l_2} \right)^3 = \frac{V_1}{V_2}, \quad \frac{l_1}{l_2} = \left( \frac{A_1}{A_2} \right)^{\frac{1}{3}} = \left( \frac{V_1}{V_2} \right)^{\frac{1}{3}}
\]

\( l_1 \) : length of figure 1, \( l_2 \) : corresponding length of figure 2
\( A_1 \) : face area of figure 1, \( A_2 \) : corresponding face area of figure 2
\( V_1 \) : volume of figure 1, \( V_2 \) : volume length of figure 2

Congruent Triangles

**Properties of congruent triangles:**
If \( \triangle ABC \cong \triangle XYZ \), then
(i) \( \angle A = \angle X, \angle B = \angle Y, \angle C = \angle Z \),
(ii) \( AB = XY, BC = YZ, CA = ZX \).

**Conditions for congruent triangles:**
(i) SSS
(ii) SAS
(iii) AAS
(iv) ASA
(v) RHS

Similar Triangles

**Properties of similar triangles:**
If \( \triangle ABC \sim \triangle XYZ \), then
(i) \( \angle A = \angle X, \angle B = \angle Y, \angle C = \angle Z \),
(ii) \( \frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX} \).

**Conditions for similar triangles:**
(i) AAA
(ii) 3 sides prop.
(iii) ratio of 2 sides, inc. \( \frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX} \)
Transformation and Symmetry in 2-D figures
Refelctional symmetry and rotational symmetry

Transformation of 2-D figures
Translation

Reflection

Rotation

Enlargement

Geometry

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Meaning</th>
<th>Abbreviation</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>adj. ∠s on st. line</td>
<td>[a + 47^\circ = 180^\circ]</td>
<td>∠s at a pt.</td>
<td>[b + 75^\circ + 135^\circ = 360^\circ]</td>
</tr>
<tr>
<td>vert. opp. ∠s</td>
<td>[c = 50^\circ]</td>
<td>corr. ∠s, (AB//CD)</td>
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<td>alt. ∠s, (AB//CD)</td>
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<td>Measures, Shape and Space</td>
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<tr>
<td>--------------------------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \angle \text{sum of } \triangle )</td>
<td>( a + b + c = 180^\circ )</td>
<td>( \text{ext. } \angle \text{of } \triangle )</td>
<td></td>
</tr>
<tr>
<td>( \angle \text{s isos. } \Delta )</td>
<td>If ( AB = AC ), then ( \angle B = \angle C )</td>
<td>( \text{sides opp. eq. } \angle \text{s} )</td>
<td></td>
</tr>
<tr>
<td>( \text{prop. of isos. } \triangle )</td>
<td>If ( AB = AC ) and one of the following conditions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{prop. of equil. } \triangle )</td>
<td>( \text{prop. of equil. } \triangle )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{Pyth. theorem} )</td>
<td>In ( \triangle ABC ), if ( \angle C = 90^\circ ), then ( a^2 + b^2 = c^2 )</td>
<td>( \text{Converse of Pyth. thm.} )</td>
<td></td>
</tr>
<tr>
<td>( \angle \text{sum of polygon} )</td>
<td>Sum of all interior angles of ( n )-sided polygon = ( (n-2)\times180^\circ )</td>
<td>( \text{sum of ext. } \angle \text{s of polygon} )</td>
<td></td>
</tr>
<tr>
<td>( \text{prop. of trapezium} )</td>
<td>( \angle P + \angle R = 180^\circ ) ( \angle Q + \angle S = 180^\circ )</td>
<td>( \text{opp. sides equal} )</td>
<td></td>
</tr>
</tbody>
</table>

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Measures, Shape and Space

<table>
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<tr>
<th>opp. $\angle$s of //gram</th>
<th>$\angle A = \angle C$ and $\angle B = \angle D$</th>
<th>diag. of //gram</th>
<th>$AO = CO$ and $BO = DO$</th>
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<tbody>
<tr>
<td>prop. of rhombus</td>
<td>prop. of //gram diagonals bisect interior angles diagonals are ⊥</td>
<td>prop. of rectangle</td>
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<td>prop. of rhombus</td>
<td>line from centre ⊥ chord bisects chord</td>
<td>If $ON \perp AB$, then $AN = BN$.</td>
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<tr>
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<td>prop of rectangle</td>
<td>equal chords, equidistant from centre</td>
<td>If $OM \perp AB$, $ON \perp CD$ and $AB = CD$, then $OM = ON$.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\angle$ at centre twice $\angle$ at $O$</td>
<td>$q = 2p$</td>
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<tr>
<td></td>
<td></td>
<td>converse of $\angle$ in semi-circle</td>
<td>If $\angle APB = 90^\circ$, then $AB$ is a diameter.</td>
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<tr>
<td></td>
<td></td>
<td>equal $\angle$s, equal arcs</td>
<td>If $\angle AOB = \angle COD$, then $\widehat{AB} = \widehat{CD}$</td>
</tr>
</tbody>
</table>

If $AB$ is a diameter, then $\angle APB = 90^\circ$. If $\angle APB = 90^\circ$, then $AB$ is a diameter.
equal ∠s, equal chords

If \( \angle AOB = \angle COD \), then \( AB = CD \)

equal arcs, equal ∠s

If \( \widehat{AB} = \widehat{CD} \), then \( \angle AOB = \angle COD \)

equal arcs, equal chords

If \( \widehat{AB} = \widehat{CD} \), then \( AB = CD \)

equal chords, equal ∠s

If \( AB = CD \), then \( \angle AOB = \angle COD \)

equal chords, equal arcs

If \( AB = CD \), then \( \widehat{AB} = \widehat{CD} \)

arcs prop. to ∠s at centre

\( \widehat{AB} : \widehat{CD} = x : y \)

arcs prop. to ∠s at \( \odot \)

\( \widehat{AB} : \widehat{CD} = m : n \)

opp. ∠s, cyclic quad.

\( \angle A + \angle C = 180^\circ \) and
\( \angle B + \angle D = 180^\circ \)

ext. ∠, cyclic quad

\( \angle DCE = \angle A \)

Converse of ∠s in the same segment

If \( p = q \), then \( A, B, C \) and \( D \) are concyclic.

opp. ∠s supp.

If \( a + c = 180^\circ \) (or \( b + d = 180^\circ \) ), then \( A, B, C \) and \( D \) are concyclic.

ext. ∠ = int. opp. ∠

If \( p = q \), then \( A, B, C \) and \( D \) are concyclic.

tangent ⊥ radius

If \( PQ \) is the tangent to the circle at \( T \), then \( PQ \perp OT \).

corverse of tangent ⊥ radius

If \( PQ \perp OT \), then \( PQ \) is the tangent to the circle at \( T \).
If two tangents, $TP$ and $TQ$, are drawn to a circle from an external point $T$ and touch the circle at $P$ and $Q$ respectively, then

(i) $TP = TQ$,
(ii) $\angle POT = \angle QOT$,
(iii) $\angle PTO = \angle QTO$.

A tangent-chord angle of a circle is equal to an angle in the alternate segment.

$\angle ATQ = \angle ABT$

If $\angle ATQ = \angle ABT$, then $PQ$ is the tangent to the circle at $T$.

**Special lines in a triangle**

$AD$ is the median of $BC$ in $\triangle ABC$.

$CD$ is the angle bisector of $\angle ACB$ in $\triangle ABC$.

$CD$ is the altitude of $AB$ in $\triangle ABC$.

$MN$ is the perpendicular bisector of $AC$ in $\triangle ABC$.

**Special points of a triangle**

Intersecting point of 3 medians
- Centroid

Intersecting point of 3 angle bisectors
- Incentre
1. These 4 points exist for any triangle. But they may or may not coincide.
2. There exists exactly one circle with in–centre as centre such that it touches all the 3 sides of the triangle. (i.e. the 3 sides are tangents.)
3. There exists exactly one circle with circumference as centre such that it passes through all the 3 vertices of the triangle.

Coordinates Geometry

Let the points $A, B$ be $(x_1, y_1)$ and $(x_2, y_2)$ respectively.

**Distance formula.**

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

mid–point of $AB = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

If $AP : PB = m : n$, then $P = \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n}\right)$

**Straight lines**

Slope of $AB = \frac{y_2 - y_1}{x_2 - x_1}$

Let $m_1, m_2$ be slopes of lines $L_1, L_2$ respectively and $L_1, L_2$ are not vertical lines. Then $L_1 \parallel L_2, m_1 = m_2, \quad L_1 \perp L_2, m_1m_2 = -1$.

**Equation of straight lines**

*Point slope form*

$$y - y_1 = m(x - x_1)$$

*General form*

$$Ax + By + C = 0$$

Slope $= \frac{-A}{B}$ if $B \neq 0$ (if $B = 0$, is undefined)

$x$–intercept $= \frac{-C}{A}$ if $A \neq 0$ (if $A = 0$, the line does not cut the $x$–axis)

$y$–intercept $= \frac{-C}{B}$ if $B \neq 0$ (if $B = 0$, the line does not cut the $y$–axis)
Equation of a circle
Let \((h, k)\) be the centre and \(r\) be the length of the radius.

Centre–radius form
\[(x - h)^2 + (y - k)^2 = r^2\]

General form
\[x^2 + y^2 + Dx + Ey + F = 0\]

Centre \[= \left(\frac{-D}{2}, \frac{-E}{2}\right)\]

Radius \[\sqrt{\left(\frac{D}{2}\right)^2 + \left(\frac{E}{2}\right)^2 - F}\]

Polar coordinates

Polar coordinates of \(P = (r, \theta)\), \(r = OP\), \(\tan \theta = \text{slope of } OP\)

Trigonometric ratios of an acute angle \(\theta\)

\[\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}}\]
\[\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}}\]
\[\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}}\]

Trigonometric ratios any angle \(\theta\)

Sign of trigonometric ratios

Trigonometric identities

1. \(\tan \theta = \frac{\sin \theta}{\cos \theta}\)
2. \(\sin^2 \theta + \cos^2 \theta = 1\)
3. \(\sin(90^\circ - \theta) = \cos \theta\)
4. \(\cos(90^\circ - \theta) = \sin \theta\)
5. \(\tan(90^\circ - \theta) = \frac{1}{\tan \theta}\)
6. \(\sin(90^\circ + \theta) = \cos \theta\)
7. \(\cos(90^\circ + \theta) = -\sin \theta\)
8. \(\tan(90^\circ + \theta) = \frac{-1}{\tan \theta}\)
9. \( \sin(180^\circ - \theta) = \sin \theta \)
10. \( \cos(180^\circ - \theta) = -\cos \theta \)
11. \( \tan(180^\circ - \theta) = -\tan \theta \)
12. \( \sin(180^\circ + \theta) = -\sin \theta \)
13. \( \cos(180^\circ + \theta) = -\cos \theta \)
14. \( \tan(180^\circ + \theta) = \tan \theta \)
15. \( \sin(-\theta) = \sin(360^\circ - \theta) = -\sin \theta \)
16. \( \cos(-\theta) = \cos(360^\circ - \theta) = \cos \theta \)
17. \( \tan(-\theta) = \tan(360^\circ - \theta) = -\tan \theta \)
18. \( \sin(360^\circ + \theta) = \sin \theta \)
19. \( \cos(360^\circ + \theta) = \cos \theta \)
20. \( \tan(360^\circ + \theta) = \tan \theta \)

**Graphs of trigonometric functions**

Graphs of \( y = \sin x \)

Graphs of \( y = \cos x \)

Graphs of \( y = \tan x \)
Measures, Shape and Space

Application of Trigonometry

Sine Formula
\[ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \]

Cosine Formula
\[ c^2 = a^2 + b^2 - 2ab\cos C \]

Area of a triangle
Area = \( \frac{1}{2}ab\sin C \)

Heron's Formula
Area = \( \sqrt{s(s-a)(s-b)(s-c)} \) where \( s = \frac{a+b+c}{2} \).

Angle between a line and a plane

A. Projection on a plane
   (a) Projection of a point on a plane

   ![Projection of a point on a plane](image)

   If \( PQ \) is perpendicular to any straight line (\( L_1 \) and \( L_2 \)) on \( \pi \) passing through \( Q \), then
   (i) \( PQ \) is perpendicular to \( \pi \),
   (ii) \( Q \) is the projection of \( P \) on \( \pi \),
   (iii) \( PQ \) is the shortest distance between \( P \) and \( \pi \).

B. Angles between lines and planes
   (a) Angle \( \theta \) between two intersecting straight lines.

   ![Angle between two intersecting straight lines](image)

   \( \theta \) is the acute angle formed by the two lines. It is called the angle of intersection.

   (c) Angle \( \theta \) between two intersecting planes
      (i) \( AB \) is the line of intersection of \( \pi_1 \) and \( \pi_2 \).
      (ii) \( \theta \) is the acute angle between two intersecting straight lines on the planes, and which are perpendicular to \( AB \).

C. Line of greatest slope
   Lines of greatest slope
   (a) \( AB \) is the line of intersection of the planes \( ABCD \) and \( ABEF \).
   (b) \( PX, L_1, L_2 \) and \( L_3 \) are perpendicular to \( AB \).
       They are lines of greatest slope of the inclined plane \( ABEF \).

Angle of elevation and angle of depression
Data Handling

Simple Statistical Diagrams and Graphs

**Pie chart**

\[
\text{Angle of the sector representing the colour ‘Blue’ } = x^\circ = 360^\circ \times 40\% = 144^\circ.
\]

**Histograms**

\[\begin{array}{|c|c|c|c|}
\hline
\text{Hourly wage} & \text{Class boundaries} & \text{Class mark} & \text{Frequency} \\
($) & ($) & ($) & \\
\hline
25 – 29 & 24.5 – 29.5 & 27 & 15 \\
30 – 34 & 29.5 – 34.5 & 32 & 22 \\
35 – 39 & 34.5 – 39.5 & 37 & 11 \\
40 – 44 & 39.5 – 44.5 & 42 & 4 \\
45 – 49 & 44.5 – 49.5 & 47 & 3 \\
\hline
\end{array}\]

**Frequency polygon**

\[\begin{array}{|c|c|c|c|}
\hline
\text{Time spent} & \text{Class mark} & \text{No. of} & \\
(min) & (min) & \text{customers} \\
\hline
11 – 20 & 15.5 & 9 \\
21 – 30 & 25.5 & 21 \\
31 – 40 & 35.5 & 16 \\
41 – 50 & 45.5 & 8 \\
51 – 60 & 55.5 & 4 \\
61 – 70 & 65.5 & 2 \\
\hline
\end{array}\]

Stem-and-leaf diagram

\[\begin{array}{|c|c|c|}
\hline
\text{Weights of 10 dogs} & \text{Stem (10 kg)} & \text{leaf (1 kg)} \\
\hline
5 & 6 \\
1 & 1 & 7 \\
3 & 4 & 6 \\
4 & 2 \\
5 & 1 \\
\hline
\end{array}\]

The data presented above are: 15 kg, 16 kg, 21 kg, 21 kg, 27 kg, 33 kg, 34 kg, 36 kg, 42 kg, 51 kg
Data Handling

Cumulative frequency polygon
The following table shows the age distribution of 50 teachers in a school

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>2</td>
<td>9</td>
<td>18</td>
<td>11</td>
<td>6</td>
<td>4</td>
</tr>
</tbody>
</table>

The cumulative frequency table of the above data is constructed below

<table>
<thead>
<tr>
<th>Age less than</th>
<th>19.5</th>
<th>24.5</th>
<th>29.5</th>
<th>34.5</th>
<th>39.5</th>
<th>44.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cumulative Frequency</td>
<td>0</td>
<td>2</td>
<td>11</td>
<td>29</td>
<td>40</td>
<td>46</td>
</tr>
<tr>
<td>Frequency</td>
<td>50</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The cumulative frequency polygon of the above data is constructed below

![Cumulative frequency polygon](image)

Measures of central tendency

**Mean**

Ungrouped mean \( \bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \)

Grouped mean \( \bar{x} = \frac{f_1 x_1 + f_2 x_2 + f_3 x_3 + \cdots + f_n x_n}{f_1 + f_2 + f_3 + \cdots + f_n} \)

Weighted mean \( \bar{x} = \frac{w_1 x_1 + w_2 x_2 + w_3 x_3 + \cdots + w_n x_n}{w_1 + w_2 + w_3 + \cdots + w_n} \)

**Median**

For ungrouped data, if number of data, \( n \), is:

(i) **odd**, median = middle datum
(ii) **even**, median = mean of the 2 middle data

For data grouped into classes, the median can be determined from its cumulative frequency polygon/curve.

e.g. The cumulative frequency polygon on the right shows the donations by a group of students to a charity fund. From the graph, the number of students is 100. From the graph, the median donation by the group of students is $52.

**Mode**

The data item with the highest frequency.

**Model class**

The class with the highest frequency
Data Handling

Measures of Dispersion

**Range**

Ungrouped data: range = largest datum – smallest datum
Grouped data: range = highest class boundary – lowest class boundary.

**Inter-quartile range**

Ungrouped data:

\[
\text{inter-quartile range} = Q_3 - Q_1
\]

Grouped data:

The quartiles can be read from the corresponding cumulative frequency polygon (or curve).

\[
\text{inter-quartile range} = Q_3 - Q_1
\]

**Box-and-whisker diagram**

The box-and-whisker diagram is an effective way to present the lower quartile, the upper quartile, the median, the maximum and the minimum values about a data set

**Standard deviation**

\[
\sigma = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n}}
\]

For grouped data,

\[
\sigma = \sqrt{\frac{f_1(x_1 - \bar{x})^2 + f_2(x_2 - \bar{x})^2 + f_3(x_3 - \bar{x})^2 + \cdots + f_n(x_n - \bar{x})^2}{f_1 + f_2 + f_3 + \cdots + f_n}}
\]

**Standard Score**

For a set of data with mean \( \bar{x} \) and standard deviation \( \sigma \), the standard score \( z \) of a given datum \( x \) is given by

\[
z = \frac{x - \bar{x}}{\sigma}
\]

**Normal distribution**

Characteristics of a normal curve:

1. It is bell-shaped.
2. It has reflectional symmetry about \( x = \bar{x} \).
3. The mean, median and mode are all equal to \( \bar{x} \).
Probability

**Theoretical probability**
In the case of an activity where all the possible outcomes are equally likely, the probability of an event \( E \), denoted by \( P(E) \), is given by

\[
P(E) = \frac{\text{number of outcomes favourable to the event } E}{\text{total number of possible outcomes}}
\]

where \( 0 \leq P(E) \leq 1 \).

Note that \( P(\text{impossible event}) = 0 \) and \( P(\text{certain event}) = 1 \)

**Experimental probability**
The experimental probability \( P(E) \) of an event \( E \) is defined as follows:

\[
P(E) = \frac{\text{number of times event } E \text{ occurs in an experiment}}{\text{total number of trials}} \quad \text{or}
\]

\[
P(E) = \text{relative frequency of event } E
\]

For a large number of trials, the experimental probability is close to the theoretical probability which is deduced from theories.

**Permutation**
An arrangement of a set of objects selected from a group in a definite order is called a permutation.

(i) The number of permutations of \( n \) distinct objects is \( n! \).
   e.g. Arrange 5 students in a row.
   Number of arrangements = \( 5! = 120 \)

(ii) The number of permutations of \( n \) distinct objects taken \( r \) at a time, denoted by \( P_r^n \), is

\[
\frac{n!}{(n-r)!}
\]

   e.g. Choose 5 out of 7 students, and arrange them in a row.
   Number of arrangements = \( P_5^7 = 2520 \)

**Combination**
A selection of \( r \) objects from \( n \) distinct objects regardless of their order is called a combination.

The number of combinations of \( n \) distinct objects taken \( r \) at a time, denoted by \( C_r^n \), is

\[
\frac{n!}{(n-r)!r!}
\]

   e.g. Choose 2 from a list of 12 friends to invite to a party.
   Number of ways of choosing = \( C_2^{12} = 66 \)

**Additional law of probability**
Mutually exclusive events – events that cannot occur at the same time..
If events \( A \) and \( B \) are mutually exclusive, then

\[
P(A \cup B) = P(A) + P(B)
\]

If events \( E_1, E_2, E_3, \ldots, E_n \) are mutually exclusive events, then

\[
P(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n) = P(E_1) + P(E_2) + P(E_3) + \cdots + P(E_n)
\]

If events \( A \) and \( B \) are not mutually exclusive, then

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{where} \quad P(A \cap B) \neq 0
\]

**Multiplication Law of Probability**
If the probability that one event occurs is not affected by the occurrence of another event, these two events are called independent events.

Multiplication law of probability for independent events:
If \( A \) and \( B \) are two independent events, then

\[
P(A \cap B) = P(A) \times P(B)
\]

If the probability that one event occurs is affected by the occurrence of another event, these two events are called dependent events.
**Data Handling**

Multiplication law of probability for dependent events:
If $A$ and $B$ are two dependent events, then $P(A \cap B) = P(A) \times P(B \mid A)$, where
$P(B \mid A)$ is the probability that event $B$ occurs given that event $A$ has occurred.

Expected value (i.e expectation)
$$E(X) = P_1 x_1 + P_2 x_2 + \cdots + P_n x_n,$$
where $P_1, P_2, \cdots$ are probabilities of events $E_1, E_2, \cdots$ and
$x_1, x_2, \cdots$ are the corresponding value of $E_1, E_2, \cdots$.

Complementary events
If $A$ and $A'$ are complementary events, then
1. $A'$ happens if $A$ does not happen and $B$ does not happen if $A$ happens;
2. $P(A \cap A') = 0$ (i.e. mutually exclusive);
3. $P(A') = 1 - P(A)$.